BENG 100. Introduction to Probability for Bioengineering

Recorded lectures

Module 1: Randomness and probability axioms

Lecture 1. Randomness and set theory

- Reading materials: Pishro-Nik chapter 1.0 – 1.2.2; Handout: Throwing a dart; Handout: Computer simulation challenge; Handout: The π challenge.
- Core concepts:
  - Randomness
  - Interpretation of probability: frequency vs. subjective personal belief
  - Set theory and set operations
  - Venn diagram
- Not required contents: Pishro-Nik chapter 1.2.2: Cartesian product.

Section 1, http://www.kaltura.com/tiny/yy2oh

1.1. Randomness

- Interpretation of probability: frequency vs. subjective personal belief; Probably is a branch of mathematics based on axioms.
- Example 1.1.1. Signal Transmission.

1.2. Set theory

- Set; the mathematical statement of a set; subset; null set; universal set.

Section 2, http://www.kaltura.com/tiny/lkywl

1.2.1. Venn diagrams.

1.2.2. Set operations.

- Union; intersection; complement; mutually exclusive sets; partition.
- Theorem 1.1: De Morgan's law;
- Theorem 1.2: Distributive law.

Lecture 2. Functions and probability axioms

- Reading materials: Pishro-Nik chapter 1.2.3 – 1.3.2; Handout: Computer simulation challenge; Handout: The π challenge.
- Videos: Video 1.2 – Sets, Functions; Video 1.3 – Probability Axioms.
- Core concepts:
  - Cardinality and finite sets
  - Inclusion-exclusion principle
  - Functions: input, output, the unique output of any input, domain, co-domain, range.
  - Random experiments: outcome, sample space, event.
  - Union and intersection of two events.
  - Axioms of probability
- Not required contents: Pishro-Nik Theorem 1.4, Theorem 1.5.

Section 1, http://www.kaltura.com/tiny/mzorg

1.2.3: Cardinality: Countable and Uncountable Sets

- Cardinality
- Inclusion-exclusion principle
- Countable and uncountable infinite sets.
- Definition 1.1.
- (Note) Theorem 1.3 – 1.5 are not discussed.

--- Please cut out 07:44 – 07:52 (“The next concept we will discuss in this course is functions.” This sentence is useless because we are going to break the video into sections).

Section 2, http://www.kaltura.com/tiny/sonuh
1.2.4: Functions
- The mapper, input, output, the unique output of any input, domain, co-domain, range.
- Example 1.6

Section 3, http://www.kaltura.com/tiny/lqg77
1.3.1. Random experiments
- Random experiment, outcome, sample space, event.
- Union and intersection of two events.
- Example 1.7.

Section 4, http://www.kaltura.com/tiny/iuyb9
1.3.2. Axioms of probability
- The definition of probability P(A) to an event A. P(A)’s input, output, domain, and range.
- The 3 Axioms of probability.

Lecture 3. Calculating probabilities
- Reading materials: Pishro-Nik chapter 1.3.2 – 1.3.4; Handout: The Monty Hall problem.
- Videos: Video 1.4 – Discrete and Continuous Probability Models.
- Important concepts:
  - Discrete probability model

Section 1, http://www.kaltura.com/tiny/tsazp
1.3.3. Calculating probabilities.
- Example 1.9

Section 2, http://www.kaltura.com/tiny/v9usn
- Example 1.10
  1. For any event A, $P(A^c)=1-P(A)$.
  2. The probability of the empty set is zero, i.e., $P(\emptyset)=0$.
  3. For any event A, $P(A)\leq 1$.

Section 3, http://www.kaltura.com/tiny/vld9h
  4. $P(A-B)=P(A)-P(A\cap B)$.
  5. $P(A\cup B)=P(A)+P(B)-P(A\cap B)$.
  6. If $A\subset B$ then $P(A)\leq P(B)$.

Section 4, http://www.kaltura.com/tiny/w5qqq
1.3.4. Discrete probability models.
1. Example 1.12.

Section 5, http://www.kaltura.com/tiny/wgay1
2. Example 1.13.

Module 2: Conditional probability, law of total probability, Bayes rule
Lecture 4. Conditional probability and independence

- Reading materials: Pishro-Nik chapter 1.4 – 1.4.1; Handout: Proof of the chain rule of conditional probability.
- Videos: Video 1.5 – Conditional Probability; Video 1.6 – Independent Events – Part 1; Video 1.7 – Independent Events – Part 2.
- Core concepts:
    - Conditional probability
    - Chain rule of conditional probability
    - Independence
    - The difference between independence and disjoint
- Not required contents: Prior probability; The proof of Lemma 1.1.

Section 1, http://www.kaltura.com/tiny/kh58v
1.4.0. Conditional probability
1. Definition
2. Example 1.15

Section 2, http://www.kaltura.com/tiny/us7is
3. Example 1.17

Section 3, http://www.kaltura.com/tiny/u4svg
4. Example 1.18: Two daughter problem

Section 4, http://www.kaltura.com/tiny/wrs68
5. Equation 1.5. and Figure 1.23 A tree diagram
6. Chain rule for conditional probability

Section 5, http://www.kaltura.com/tiny/ucl8y
1.4.1. Independence.
1. Definition
2. Example 1.20

Section 6, http://www.kaltura.com/tiny/rp1cs
3. Example 1.21
4. Lemma 1.1

Section 7, http://www.kaltura.com/tiny/n8vbq
5. The difference btw Independence and disjoint, and Lemma 1.2.

Lecture 5. Law of total probability

- Reading materials: Pishro-Nik chapter 1.4.2; Handout: The Monty Hall Problem.
- Videos: Video 1.8 – Law of Total Probability.
- Core concepts:
    - Law of Total Probability

Section 1, http://www.kaltura.com/tiny/i5hbm
1.4.2. Law of total probability
1. Deriving the law of total probability

Section 2, http://www.kaltura.com/tiny/r137k

2. Example 1.24

Section 3, http://www.kaltura.com/tiny/tdqgn

3. Handout the Monty Hall problem

Lecture 6. Bayes rule

- Reading materials: Pishro-Nik chapter 1.4.3.
- Videos: Video 1.9 – Bayes Theorem, Video 1.10 – Boy or Girl Paradox.
- Core concepts:
  - Bayes rule
- Not required contents: Pishro-Nik chapter 1.4.4: Conditional independence.

Section 1, http://www.kaltura.com/tiny/x2yrh

1.4.3. Bayes Rule

1. Deriving the Bayes rule

Section 2, http://www.kaltura.com/tiny/locql

2. Example 1.25

Section 3, http://www.kaltura.com/tiny/yzwwu

3. Example 1.26: false positive paradox.

Section 4, http://www.kaltura.com/tiny/raion

4. Review of Module 2: Conditional probability, law of total probability, Bayes rule

Module 3: Combinatorics

Lecture 7. Sampling, permutations, and combinations

- Reading materials: Pishro-Nik chapter 2.1 – 2.1.2.
- Videos: Video 2.1 – Counting – Part 1; Video 2.2 – Counting – Part 2.
- Core concepts:
  - Ordered sampling with replacement
  - Permutations: Ordered sampling without replacement

Section 1, http://www.kaltura.com/tiny/y4lr7

1. Example 2.1
2. Multiplication principle
3. Introduction to sampling
    1. Sampling with replacement.
    2. Sampling without replacement.
    3. Ordered and unordered sampling.

Section 2, http://www.kaltura.com/tiny/ybr5p

2.1.1. Ordered sampling with replacement.

Section 3, http://www.kaltura.com/tiny/kl73w

2.1.2. Permutations

Section 4, http://www.kaltura.com/tiny/mfhqr

1. Example 2.4. The Birthday problem.

Section 5, http://www.kaltura.com/tiny/z5rxs

2.1.3. Combinations

1. Example 2.7

Section 6, http://www.kaltura.com/tiny/xa268

2. Example 2.6

Lecture 8. Combinations, binomial distribution.

- Reading materials: Pishro-Nik chapter 2.1.3.
- Textbook videos: Video 2.3 - Counting - Part 3: K-Combinations and the Binomial Formula.
- Core concepts:
  - Combinations: Unordered sampling without replacement
  - Bernoulli Trials
  - Binomial Distribution
- Not required contents: Pishro-Nik chapter 2.1.4: Unordered sampling with replacement.

Section 1, http://www.kaltura.com/tiny/wdh7q

2.1.3. Combinations.

1. Bernoulli trials and Binomial formula

Section 2, http://www.kaltura.com/tiny/tur4m

2. Example 2.9

Module 4: Random variables and discrete distributions

Lecture 9. Random variables

- Reading materials: Pishro-Nik chapter 3.1 – 3.1.4.
- Textbook videos: Video 3.1 – Introduction to Random Variables: Discrete Random Variables - Part 1; Video 3.2 - Discrete Random Variables, PMF, Independent Random Variables.
- Core concepts:
  - Random variable (RV)
  - Discrete random variable
  - Probability mass function (PMF)
  - Independent random variables

Section 1, http://www.kaltura.com/tiny/jrwgr

3.1.1. Random variables.

1. Definition.
2. Example 3.1.
3. Example 3.2.

Section 2, http://www.kaltura.com/tiny/v6nf7

3.1.3. Probability mass function

1. Definition
2. Example 3.3
3. Plotting a PMF: Figure 3.1

Section 3, http://www.kaltura.com/tiny/uztwv

4. Example 3.4

Section 4, http://www.kaltura.com/tiny/ijulz

3.1.4 Independent random variables

1. Definition 3.2.
2. Example 3.6.
3. Definition 3.3.

Lecture 10. Bernoulli, Geometric, and binomial distributions.

- Reading materials: Pishro-Nik chapter 3.1.5.
- Textbook videos: Video 3.3 – Geometric and Binomial Random Variables.
- Core concepts:
  - Bernoulli distribution
  - Geometric distribution
  - Binomial distribution
- Not required contents: Pishro-Nik chapter 2.1.5: Negative Binomial distribution.

Section 1, http://www.kaltura.com/tiny/ynzfy

3.1.5. Special distributions

1. Bernoulli distribution: Figure 3.2

Section 2, http://www.kaltura.com/tiny/svus2

2. Geometric distribution: Figure 3.3

Section 3, http://www.kaltura.com/tiny/z45fq

3. Binomial distribution: Figures 3.4, 3.5

Section 4, http://www.kaltura.com/tiny/r0109

4. Binomial random variable as a sum of Bernoulli random variables (Lemma 3.1)

3.1.6. Solved problems: Problem 2.

Section 5, http://www.kaltura.com/tiny/zapo8

3.1.6. Solved problems: Problem 3

Lecture 11. Hypergeometric distribution and applications to genetic data

- Reading materials: Pishro-Nik chapter 3.1.5; Handout: Applications of the hypergeometric distribution: Color of cards; Patients in a clinic; Handout: Genotype vs. phenotype
- Textbook videos: Video 3.4 – Poisson, Pascal, and Hypergeometric Distributions
- Core concepts:
  - Hypergeometric distribution
  - Genotype
  - Phenotype
  - Single nucleotide polymorphism (SNP)
- Not required contents: Pishro-Nik chapter 2.1.5: Poisson distribution.

Section 1, http://www.kaltura.com/tiny/tmjxc
3.1.5 Introduction of Hypergeometric distribution.

Section 2, http://www.kaltura.com/tiny/lyfsm
Handout: Applications of the Hypergeometric distribution
1. Solving the Color of cards problem from the handout.
2. Solving the Patients in a clinic problem from the handout.

Section 3, http://www.kaltura.com/tiny/m7vgo
Handout: Introduction to Genotype and Phenotype.
1. Solving the Genotype vs. cancer problem from the handout.

Module 5: Cumulative distribution function, functions of RV, continuous RV.

Lecture 12. Cumulative distribution function, Expectation

- Reading materials: Pishro-Nik chapter 3.2.1 – 3.2.2.
- Textbook videos: Video 3.5 – CDF for Discrete Random Variables, Video 3.6 – Expectation of Discrete Random Variables.
- Core concepts:
  - Cumulative distribution function (CDF)
  - Expectation
- Not required contents: Pishro-Nik chapter 3.2.2: Example 3.13.

Section 1, http://www.kaltura.com/tiny/rd96x

3.2.1. Cumulative distribution function
1. Definition 3.10

Section 2, http://www.kaltura.com/tiny/qyl0d

2. Example 3.9: Figure 3.3

Section 3, http://www.kaltura.com/tiny/lr86j

3. The relationship between CDF and PDF for discrete distributions: Figure 3.4

Section 4, http://www.kaltura.com/tiny/juls9

4. Expressing the CDF as a sum of PMF: Equation 3.1

Section 5, http://www.kaltura.com/tiny/i7npq

5. Example 3.10

Section 6, http://www.kaltura.com/tiny/y6j5n

3.2.2. Expectation
1. Definition 3.11
2. Example 3.11: E(X) for a Bernoulli RV.

Section 7, http://www.kaltura.com/tiny/yjf8u

3. Theorem 3.2: Expectation of sum of RVs.
4. Example 3.14. Expectation of a Binomial RV.

Lecture 13. Functions of random variables, Variance

- Reading materials: Pishro-Nik chapter 3.2.3 – 3.2.4.
- Textbook videos: Video 3.8 – Functions of Discrete Random Variables, Video 3.9 - Variance and Standard Deviation.
- Core concepts:
    - Function of RV
    - Expected value of the function of a RV
    - Variance
    - Variance of the sum of independent RVs

Section 1, http://www.kaltura.com/tiny/qgubb

3.2.3. Functions of random variables

1. Definition
2. Example 3.16, Y= 2|X|, PMF (Y)

Section 2, http://www.kaltura.com/tiny/yqybc

- Expected value of the function of a RV (LOTUS)

Section 3, http://www.kaltura.com/tiny/mkgig

- Example 3.17

Section 4, http://www.kaltura.com/tiny/lile3

3.2.4. Variance

1. Definition
2. Example from Equations 3.3 and 3.4

Section 5, http://www.kaltura.com/tiny/ylh5k

3. Standard deviation
4. Equation 3.5

Section 6, http://www.kaltura.com/tiny/jewc7

- Example 3.19

Section 7, http://www.kaltura.com/tiny/voeps

- Theorem 3.3: Variance of the linear transformation of a RV
- Theorem 3.4: Variance of the sum of independent RVs

Section 8, http://www.kaltura.com/tiny/zmw51

- Example 3.20: Variance of a Binomial RV

Module 6: Continuous RV, uniform and normal distributions, law of large numbers

Lecture 14. Functions of random variables, Variance

- Reading materials: Pishro-Nik chapter 4.1.0 – 4.1.2.
- Textbook videos: Video 4.2 – Probability Density Function (PDF) for Continuous Random Variables; Video 4.3 - Expected Values for Continuous Random Variables
- Core concepts:
  - Probability density function (PDF)
  - Expectation and variance
  - Law of the unconscious statistician (LOTUS)
- Not required contents: Pishro-Nik chapter 4.1.3.

Section 1, http://www.kaltura.com/tiny/qrur3

4.1.0. Continuous random variables

1. Example 4.1: a uniform RV
2. Definition 4.1

Section 2, http://www.kaltura.com/tiny/pa3i4

4.1.1. Probability density function

1. Definition 4.2.
2. Example: the PDF of a uniform RV

Section 3, http://www.kaltura.com/tiny/ubhd1

3. Integration of PDF

Section 4, http://www.kaltura.com/tiny/pdshs

4. Example 4.2: calculating probabilities for an exponential PDF.

Section 5, http://www.kaltura.com/tiny/q7e8y

4.1.2. Expectation and variance

1. Definition of E(X)
2. Example 4.3
3. Example 4.4

Section 6, http://www.kaltura.com/tiny/qpyz4

4. Equation 4.3 (LOTUS)
5. Example 4.5

Section 7, http://www.kaltura.com/tiny/rev5k

6. Variance
7. Linear transformation of Var: Equation 4.4
8. Example 4.6

Lecture 15. Uniform distribution, Normal distribution

- Reading materials: Pishro-Nik chapter 4.2.1, 4.2.3.
- Textbook videos: Video 4.7 – Uniform Distribution, Video 4.9 – Normal distribution.
- Prior knowledge from calculus:
  - Gaussian integral (https://en.wikipedia.org/wiki/Gaussian_integral),

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

  - Integration by parts (https://en.wikipedia.org/wiki/Integration_by_parts),

$$\int_a^b u(x)v'(x)dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)dx$$

$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx.$$

- Core concepts:
  - Uniform distribution
  - Standard normal RV
  - The Φ function (CDF of standard normal)
  - Normal RVs
- Not required contents: Pishro-Nik chapter 4.2.2. Exponential distribution, 4.2.4. Gamma distribution, 4.3. Mixed RVs, 5. Joint distributions, 6. Multiple RVs.

Section 1, http://www.kaltura.com/tiny/we3z3

4.2. Special distributions

1. Uniform distribution: PDF, E(X), Var(X)

Section 2, http://www.kaltura.com/tiny/mkvtk

2. Introduction of the standard normal distribution: PDF. Demonstrating that the integration of PDF on (-inf, +inf) = 1.

Section 3, http://www.kaltura.com/tiny/sylso

3. E(Z), Var(Z)

Section 4, http://www.kaltura.com/tiny/oqvx5

4. CDF and its properties: Figure 4.7

Section 5, http://www.kaltura.com/tiny/x7v3b

5. Normal RV: X=σZ+u, F(X), f(X)
6. Using a programming language such as MATLAB to calculate P(a<X<b) for the normal RV X.

Section 6, http://www.kaltura.com/tiny/jjwwt

7. Example 4.11

Section 7, http://www.kaltura.com/tiny/i8dzf

8. Theorem 4.3.

Lecture 16. Law of large numbers

- Reading materials: Pishro-Nik chapter 7.1.1 – 7.1.2.
- Textbook videos: Video 3.8 – Functions of Discrete Random Variables, Video 3.9 - Variance and Standard Deviation.
- Core concepts:
  - Sample mean
  - The weak law of large numbers
  - Central limit theorem
- Not required contents: Pishro-Nik chapter 7.1.1: Proof of the weak law of large numbers.

Section 1, http://www.kaltura.com/tiny/tjecf

7.1.1. Law of large numbers
1. Definition 7.1: sample mean
2. $E(\overline{X}), Var(\overline{X})$
3. The weak law of large numbers

Section 2, http://www.kaltura.com/tiny/m09qn

7.1.2. Central limit theorem
1. The CLT

Section 3, http://www.kaltura.com/tiny/u0do9

2. An alternative view of the CLT
3. Example: sum of Bernoulli RVs, Figure 7.1

Section 4, http://www.kaltura.com/tiny/vfxy8

4. Example: sum of Uniform RVs.

Section 5, http://www.kaltura.com/tiny/o2oo6

1. How to apply CLT

Section 6, http://www.kaltura.com/tiny/sing0

2. Example 7.1. Important: the distribution form is unknown in this example!

Module 7: Hypothesis testing

Lecture 17. Hypothesis testing

- Reading materials: Pishro-Nik chapter 8.4.1-8.4.2, chapter 8.4.4. Handout: The instructor's cheat sheet to hypothesis testing.
- Core concepts:
  - Null and alternative hypotheses
  - Test statistic
  - P-value
- Not required contents: Pishro-Nik chapters 8.1 – 8.3, 8.4.3.

Section 1, http://www.kaltura.com/tiny/wy9yi

An overview of the major steps of hypothesis testing

Section 2, http://www.kaltura.com/tiny/kg7s7

1. Forming two competing hypotheses, called **the null (H0) and the alternative hypothesis (H1)**.

Section 3, http://www.kaltura.com/tiny/p7tsn

2. Generating or getting data.

Section 4, http://www.kaltura.com/tiny/xk0xy

3. Summarizing the data into a **Test Statistic**.

Section 5, http://www.kaltura.com/tiny/z5v05

4. Calculating the **p-value**.

Section 6, http://www.kaltura.com/tiny/jb72p

5. Making a decision based on p-value.

Lecture 18. Acceptance and rejection regions

- Reading materials: Pishro-Nik chapters 8.4.2 and 8.4.4. Handout: The instructor's cheat sheet to hypothesis testing.
- Core concept:
  - Acceptance region and rejection region
  - Type I error
  - Significance level
  - Type II error
- Not required contents: Pishro-Nik chapter 8.4.5.

(Optional) Section 1, http://www.kaltura.com/tiny/m0d8p

1. Review of the major steps for hypothesis testing

Section 2, http://www.kaltura.com/tiny/ms7th

2. An alternative way of making a decision:
- Acceptance region

Section 3, http://www.kaltura.com/tiny/ya57q

- Type I error
- Type II error
- Significance level

Section 4, http://www.kaltura.com/tiny/ptu31

3. Chapter 8.4.4, Example 8.29: Finding the acceptance region and calculating p-value.